How to Gamble If You Must

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Abstract

In *red and black*, a player bets, at even stakes, on a sequence of independent games with success probability p, until she either reaches a fixed goal or is ruined. In this article we explore two strategies: timid play in which the gambler makes the minimum bet on each game, and bold play in which she bets, on each game, her entire fortune or the amount needed to reach the target (whichever is smaller). We study the success probability (the probability of reaching the target) and the expected number of games played, as functions of the initial fortune. The mathematical analysis of bold play leads to some exotic and beautiful results and unexpected connections with dynamical systems. Our exposition (and the title of the article) are based on the classic book *Inequalities for Stochastic Processes; How to Gamble if You Must*, by Lester E. Dubbins and Leonard J. Savage.

1 Red and Black

This article will explore strategies for one of the simplest gambling models. Yet in spite of the simplicity of the model, the mathematical analysis leads to some beautiful and sometimes surprising results that have importance and application well beyond gambling. The exposition (and the title of the article) are based primarily on the classic book *Inequalities for Stochastic Processes; How to Gamble if You Must*, by Lester E. Dubbins and Leonard (Jimmie) Savage [1]. The article and applets are adapted from material in *Virtual Laboratories in Probability and Statistics* [4].

Assumptions and Random Processes

Here is the basic situation: The gambler starts with an initial sum of money. She bets on independent, probabilistically identical games, each with two outcomes–win or lose. If she wins a game, she receives the amount of the bet on that game; if she loses a game, she must pay the amount of the bet. Thus, the gambler plays at *even stakes*. This particular situation (independent, identical games and even stakes) is known as *red and black* and is named after the casino game roulette. Other examples are the *pass* and *don't pass* bets in craps.

Let us try to formulate the gambling experiment mathematically. First, let I_n denote the outcome of the nth game for $n \in \mathbb{N}_+$ where 1 denotes a win and 0 denotes a loss. These are independent indicator random variables with the same distribution:

$$\mathbb{P}(I_n = 1) = p, \quad \mathbb{P}(I_n = 0) = q := 1 - p$$

where $p \in [0, 1]$ is the probability of winning an individual game. Thus, $\boldsymbol{I} = (I_1, I_2, \ldots)$ is a sequence of *Bernoulli trials*, named for the Swiss mathematician Jacob Bernoulli. Bernoulli trials form one of the simplest yet still most important of all random processes.

If $p = 0$ then the gambler always loses and if $p = 1$ then the gambler always wins. These trivial cases are not interesting, so we will usually assume that $0 < p < 1$. In real gambling houses, of course, $p < \frac{1}{2}$ (that is, the games are unfair to the player), so we will be particularly interested in this case.

The gambler's fortune over time is the basic random process of interest: Let X_0 denote the gambler's initial fortune and X_n the gambler's fortune after n games. The gambler's strategy consists of the decisions of how much to bet on the various games and when to quit. Let Y_n denote the amount of the nth bet, and let N denote the (generally random) number of games played by the gambler. If we want to, we can always assume that the games go on forever, but with the assumption that the gambler bets 0 on all games after N. With this understanding, the game outcome, fortune, and bet processes are defined for all n.

Exercise 1. Show that the fortune process is related to the wager process as follows:

$$X_n = X_{n-1} + (2I_n - 1)Y_n, \quad n \in \mathbb{N}_+$$

Strategies

The gambler's strategy can be very complicated. For example, the random variable Y_n, the gambler's bet on game n, or the event $N = n - 1$, her decision to stop after $n-1$ games, could be based on the entire past history of the game, up to time n. This history is the vector of random variables

$$H_n = (X_0, Y_1, I_1, Y_2, I_2, \ldots, Y_{n-1}, I_{n-1})$$

Moreover, her decisions could have additional sources of randomness. For example a gambler playing roulette could partly base her bets on the roll of a lucky die that she keeps in her pocket. However, the gambler cannot see into the future (unfortunately from her point of view), so we can at least assume that Y_n and $\{N = n - 1\}$ are independent of $(I_n, I_{n+1}, \ldots)$

At least in terms of expected value, any gambling strategy is futile if the games are unfair.

Exercise 2. Use the result of Exercise 1 and the assumption of no prescience to show that

$$\mathbb{E}(X_n) = \mathbb{E}(X_{n-1}) + (2p-1)\,\mathbb{E}(Y_n), \quad n \in \{1, 2, \ldots\}$$

Exercise 3. Suppose that the gambler has a positive probability of making a real bet on game n, so that $\mathbb{E}(Y_n > 0)$. Use the result of Exercise 2 to show that

a $\mathbb{E}(X_n) < \mathbb{E}(X_{n-1})$ if $p < \frac{1}{2}$

b $\mathbb{E}(X_n) > \mathbb{E}(X_{n-1})$ if $p > \frac{1}{2}$

c $\mathbb{E}(X_n) = \mathbb{E}(X_{n-1})$ if $p = \frac{1}{2}$

Exercise 3 shows that on any game in which the gambler makes a positive bet, her expected fortune strictly decreases if the games are unfair, remains the same if the games are fair, and strictly increases if the games are favorable.

As noted earlier, a general strategy can depend on the past history and can be randomized. However, since the underlying Bernoulli games are independent, one might guess that these complicated strategies are no better than simple strategies in which the amount of the bet and the decision to stop are based only on the gambler's current fortune. These simple strategies do indeed play a fundamental role and are referred to as *stationary, deterministic strategies*. Such a strategy can be described by a betting function S from the space of fortunes to the space of allowable bets, so that $S(x)$ is the amount that the gambler bets when her current fortune is x.

For a stationary, deterministic strategy, the critical insight is that after each game, the fortune process simply starts over again, but with a different initial value. This is an example of the *Markov property*, named for Andrey Markov. Thus, our fortune process is a Markov chain, one of the most important classes of stochastic processes.

The Stopping Rule

From now on, we will assume that the gambler's stopping rule is a very simple and standard one: she will bet on the games until she either loses her entire fortune and is ruined or reaches a fixed target fortune a:

$$N = \min\{n \in \mathbb{N} : X_n = 0 \text{ or } X_n = a\}$$

Thus, any strategy (betting function) S must satisfy $S(x) \leq \min\{x, a-x\}$ for $0 \leq x \leq a$ the gambler cannot bet what she does not have, and will not bet more than is necessary to reach the target a.

If we want to, we can think of the difference between the target fortune and the initial fortune as the entire fortune of the house. With this interpretation, the player and the house play symmetric roles, but with complementary win probabilities: play continues until either the player is ruined or the house is ruined. The random variables of primary interest are N, the number of games played, and the final fortune X_N of the gambler. Note that the second variable takes just two values; 0 and a.

Exercise 4. Show that the mean and variance of the final fortune are given by

a $\mathbb{E}(X_N) = a\,\mathbb{P}(X_N = a)$

b $\mathrm{Var}(X_N) = a^2\,\mathbb{P}(X_N = a)[1 - \mathbb{P}(X_N = a)]$

Presumably, the gambler would like to maximize the probability of reaching the target fortune. Is it better to bet small amounts or large amounts, or does it not matter? How does the optimal strategy, if there is one, depend on the initial fortune, the target fortune, and the game win probability?

We are also interested in $\mathbb{E}(N)$, the expected number of games played. Perhaps a secondary goal of the gambler is to maximize the expected number of games that she gets to play. (Maybe she gets free drinks!) Are the two goals compatible or incompatible? That is, can the gambler maximize both her probability of reaching the target and the expected number of games played, or does maximizing one quantity necessarily mean minimizing the other?

In the next two sections, we will analyze and compare two strategies that are in a sense opposites:

Timid Play On each game, until she stops, the gambler makes a small constant bet, say \$1.

Bold Play On each game, until she stops, the gambler bets either her entire fortune or the amount needed to reach the target fortune, whichever is smaller.

In the section following the discussion of these specific strategies, we will return to the question of optimal strategies.

2 Timid Play

Recall that with the strategy of timid play in red and black, the gambler makes a small constant bet, say \$1, on each game until she stops. Thus, on each game, the gambler's fortune either increases by 1 or decreases by 1, until the fortune reaches either 0 or the target a (which we assume is a positive integer). Thus, the fortune process $(X_0, X_1, \ldots)$ is a random walk on the fortune space $\{0, 1, \ldots, a\}$ with 0 and a as absorbing barriers. The state graph is given in Figure 1. Random walks are particularly simple and important types of Markov chains.

The Probability of Winning

Our analysis based on the Markov property suggests that we treat the initial fortune as a variable. Soon we will vary the target fortune as well. Thus, we

Figure 1: The state graph for the fortune process under timid play.

will denote the probability that the gambler reaches the target a, starting with an initial fortune x by

$$f(x,a) = \mathbb{P}(X_N = a | X_0 = a), \quad x \in \{0, 1, \ldots, a\}$$

Recall that f is known as the *success function.*

Exercise 5. By conditioning on the outcome of the first game, show that f satisfies the difference equation in part (a). Show directly that f satisfies the boundary conditions in part (b).

a $f(x,a) = qf(x-1,a) + pf(x+1,a)$ for $x \in \{1, 2, \ldots, a-1\}$

b $f(0,a) = 0$, $f(1,a) = 1$

The difference equation in Exercise 5 is *linear* (in the unknown function f), *homogeneous* (because each term involves the unknown function f), has *constant coefficients* (because the factors multiplying f are constants), and *second order* (because 2 is the difference between the largest and smallest values of the initial fortune).

Exercise 6. Show that the characteristic equation of the difference equation in Exercise 5 is $pr^2 - r + q = 0$, and that the roots are $r = 1$ and $r = q/p$

Exercise 7. Show that if $p \neq \frac{1}{2}$, then the roots are distinct. Show that, in this case, the success function is

$$f(x,a) = \frac{(q/p)^{x-1}}{(q/p)^{a-1}}, \quad x \in \{0, 1, \ldots, a\}$$

Exercise 8. Show that if $p = \frac{1}{2}$, the characteristic equation has a single root 1 that has multiplicity 2. Show that, in this case, the success function is simply the ratio of the initial fortune to the target fortune:

$$f(x,a) = \frac{x}{a}, \quad x \in \{0, 1, \ldots, a\}$$

Thus, we have the distribution of the final fortune X_N in all cases:

$$\mathbb{P}(X_N = 0 | X_0 = x) = 1 - f(x,a), \quad x \in \{0, 1, \ldots, a\}$$
$$\mathbb{P}(X_N = a | X_0 = x) = f(x,a), \quad x \in \{0, 1, \ldots, a\}$$

Exercise 9. Show that as a function of x, for fixed p and a,

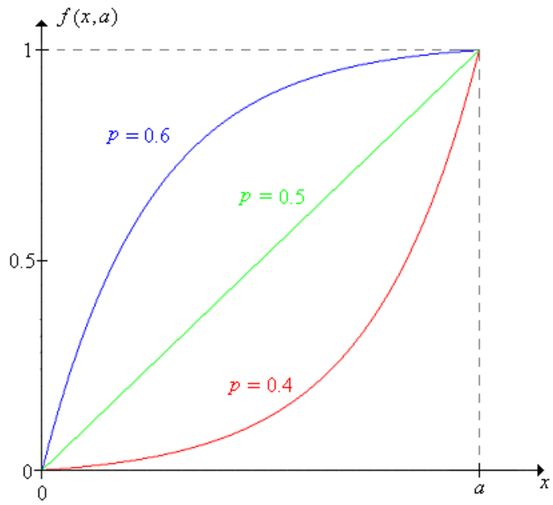

Figure 2: The graph of f for various values of p.

a $f(x, a)$ increases from 0 to 1 as x increases from 0 to a.

b f is concave upward if $p < \frac{1}{2}$ and concave downward if $p > \frac{1}{2}$. Of course, f is linear if $p = \frac{1}{2}$

Exercise 10. Show that $f(x, a)$ is continuous as a function of p, for fixed x and a. Specifically, use l'Hospital's Rule to show that the expression in Exercise 7 converges to the expression in Exercise 8, as $p \to \frac{1}{2}$.

Exercise 11. Show that for fixed x and a, $f(x, a)$ increases from 0 to 1 as p increases from 0 to 1.

The Expected Number of Trials

Now let us consider the expected number of games needed with timid play, when the initial fortune is x:

$$g(x, a) = \mathbb{E}(N|X_0 = x), \quad x \in \{0, 1, \ldots, a\}$$

Exercise 12. By conditioning on the outcome of the first game, show that g satisfies the difference equation in part (a). Show directly that g satisfies the boundary conditions in part (b).

a $g(x, a) = qg(x - 1, a) + pg(x + 1, a) + 1$ for $x \in \{1, 2, \ldots, a - 1\}$

b $g(0, a) = 0$, $g(a, a) = 0$

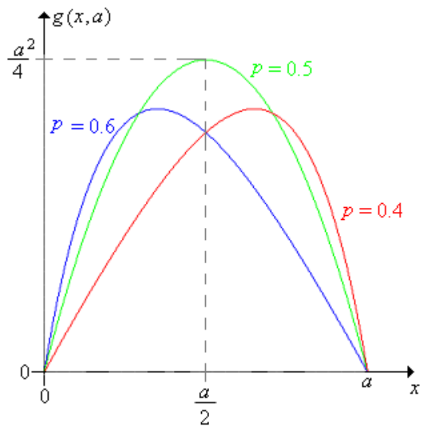

Figure 3: The graph of g for various values of p

The difference equation in the Exercise 12 is linear, second order, but *non-homogeneous* (because of the constant term 1 on the right side). The corresponding homogeneous equation is the equation satisfied by the success probability function f, given in Exercise 5. Thus, only a little additional work is needed here.

Exercise 13. Show that if $p \neq \frac{1}{2}$ then

$$g(x,a) = \frac{x}{q-p} - \frac{a}{q-p} f(x,a), \quad x \in \{0,1,\ldots,a\}$$

where f is the success function given in Exercise 7

Exercise 14. Show that if $p = \frac{1}{2}$, then

$$g(x,a) = x(a-x), x \in \{0,1,\ldots,a\}$$

Exercise 15. Consider g as a function of the initial fortune x, for fixed values of the game win probability p and the target fortune a.

a Show that g at first increases and then decreases.

b Show that g is concave downward.

Except when $p = \frac{1}{2}$, the value of x where the maximum occurs is rather complicated.

Exercise 16. Show that $g(x,a)$ is continuous as a function of p, for fixed x and a. Specifically, show that the expression in Exercise 16 converges to the expression in Exercise 14 as $p \to \frac{1}{2}$.

For many parameter settings, the expected number of games is surprisingly large. For example, suppose that $p = \frac{1}{2}$ and the target fortune is 100. If the gambler's initial fortune is 1, then the expected number of games is 99, even though half of the time, the gambler will be ruined on the first game. If the initial fortune is 50, the expected number of games is 2500.

Increasing the Bet

What happens if the gambler makes constant bets, but with an amount higher than 1? The answer to this question may give insight into what will happen with bold play.

Fix p and suppose that the target fortune is $2a$ and the initial fortune is $2x$. If the gambler plays timidly (betting \$1 each time), then of course, her probability of reaching the target is $f(2x, 2a)$. On the other hand:

Exercise 17. Suppose that the gambler bets \$2 on each game. Argue that the fortune process $(X_i/2 : i \in \mathbb{N})$ corresponds to timid play with initial fortune x and target fortune a and that therefore the probability that the gambler reaches the target is $f(x, a)$.

Thus, we need to compare the probabilities $f(2x, 2a)$ and $f(x, a)$.

Exercise 18. Show that $f(2x, 2a) = f(x, a)\frac{(q/p)^x+1}{(q/p)^a+1}$ for $x \in \{0, 1, \ldots, a\}$ and hence

a $f(2x, 2a) < f(x, a)$ if $p < \frac{1}{2}$

b $f(2x, 2a) = f(x, a)$ if $p = \frac{1}{2}$

c $f(2x, 2a) > f(x, a)$ if $p > \frac{1}{2}$

Thus, it appears that increasing the bets is a good idea if the games are unfair, a bad idea if the games are favorable, and makes no difference if the games are fair.

What about the expected number of games played? It seems almost obvious that if the bets are increased, the expected number of games played should decrease, but a direct analysis using Exercise 13 is harder than one might hope (try it!), We will use a different method, one that actually gives better results. Specifically, we will have the \$1 and \$2 gamblers bet on the same underlying sequence of games, so that the two fortune processes are defined on the same probability space. Then we can compare the actual random variables (the number of games played), which in turn leads to a comparison of their expected values. This general method is sometimes referred to as *coupling*.

Assume again that the initial fortune is $2x$ and the target fortune $2a$ where $0 < x < a$. Let X_n denote the fortune after n games for the gamble making \$1 bets (simple timid play), so that $2X_n - X_0$ is the fortune after n games for the gambler making \$2 bets (with the same initial fortune, betting on the same sequence of games). Let N_1 denotes the number of games played by the \$1 gambler, and N_2 denotes the number of games played by the \$2 gambler,

Exercise 19. Argue that

a If the \$1 gambler falls to fortune x, the \$2 gambler is ruined (fortune 0).

b If the \$1 gambler hits fortune $x + a$, the \$2 gambler is reaches the target $2a$.

c The \$1 gambler must hit x before hitting 0 and must hit $x + a$ before hitting $2a$

d $N_2 < N_1$ given $X_0 = 2x$

e $\mathbb{E}(N_2|X_0 = 2x) < \mathbb{E}(N_1|X_0 = 2x)$

Of course, the expected values agree (and are both 0) if $x = 0$ or $x = a$.

Exercise 19 shows that N_2 is *stochastically smaller* than N_1 even when the gamblers are not playing the same sequence of games (so that the random variables are not defined on the same probability space).

Exercise 20. Generalize the analysis in this subsection to compare timid play with the strategy of betting \$$k$ on each game (let the initial fortune be kx and the target fortune ka

It appears that with unfair games, the larger the bets the better, at least in terms of the probability of reaching the target. Thus, we are naturally led to consider bold play.

3 Bold Play

Recall that with the strategy of bold play in red and black, the gambler on each game bets either her entire fortune or the amount needed to reach the target fortune, whichever is smaller. As usual, we are interested in the probability that the player reaches the target and the expected number of trials. The first interesting fact is that only the ratio of the initial fortune to the target fortune matters, quite in contrast to timid play.

Exercise 21. Suppose that the gambler plays boldly with initial fortune x and target fortune a. As usual, let $\boldsymbol{X} = (X_0, X_1, \ldots)$ denote the fortune process for the gambler. Argue that for any $c > 0$, the random process $c\boldsymbol{X} = (cX_0, cX_1, \ldots)$ is the fortune process for bold play with initial fortune cx and target fortune ca.

Because of this result, it is convenient to use the target fortune as the monetary unit and to allow irrational, as well as rational, initial fortunes. Thus, the fortune space is $[0, 1]$. Sometimes in our analysis we will ignore the states 0 or 1; clearly there is no harm in this because in these states, the game is over.

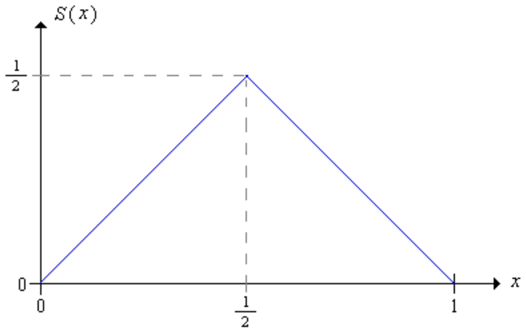

Figure 4: The betting function under bold play

Exercise 22. Recall that the betting function S is the function that gives the amount bet as a function of the current fortune. Show that

$$S(x) = \min\{x, 1-x\} = \begin{cases} x, & x \in [0, \frac{1}{2}] \\ 1-x, & x \in [\frac{1}{2}, 1] \end{cases}$$

The Probability of Winning

We will denote the probability that the bold gambler reaches the target $a = 1$ starting from the initial fortune $x \in [0,1]$ by $F(x)$. By Exercise 21, the probability that the bold gambler reaches some other target value a, starting from $x \in [0, a]$ is $F(x/a)$

Exercise 23. Exercise 3.3. By conditioning on the outcome of the first game, show that F satisfies the functional equation in part (a). Show directly that F satisfies the boundary conditions in part (b):

a $F(x) = \begin{cases} pF(2x), & x \in [0, \frac{1}{2}] \\ p + qF(2x-1), & x \in [\frac{1}{2}, 1] \end{cases}$

b $F(0) = 0$, $F(1) = 1$

The functional equation in Exercise 23 is highly nonlinear, in contrast to the corresponding result for timid play. In fact, it's clear that an important role is played by the function d defined on $[0,1]$ by

$$d(x) = 2x - \lfloor x \rfloor = \begin{cases} 2x, & x \in [0, \frac{1}{2}) \\ 2x-1, & x \in [\frac{1}{2}, 1) \end{cases}$$

The function d is sometimes called the *doubling function*, mod 1, since $d(x)$ gives the fractional part of $2x$. Note that until the last bet that ends the game

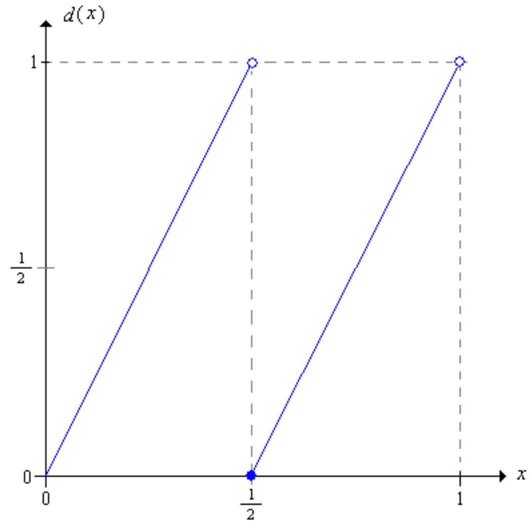

Figure 5: The doubling function mod 2

(with the player ruined or victorious), the successive fortunes of the player follow iterates of the map d. Thus, bold play is intimately connected with the *dynamical system* associated with d.

Binary Expansions

One of the keys to our analysis is to represent the initial fortune in binary form. The binary expansion of $x \in [0,1]$ is

$$x = \sum_{i=1}^{\infty} \frac{x_i}{2^i}$$

where $x_i \in \{0,1\}$ for each $i \in \mathbb{N}_+$. This representation is unique except when x is a *binary rational* (sometimes also called a *dyadic rational*), that is, a number of the form k/x^n where $n \in \mathbb{N}_+$ and $k \in \{1, 3, \ldots, 2n-1\}$; the positive integer n is called the *rank* of x. For a binary rational x of rank n, we will use the standard terminating representation where $x_n = 1$ and $x_i = 0$ for $i > n$. Rank can be extended to all numbers in $[0,1)$ by defining the rank of 0 to be 0 (0 is also considered a binary rational) and by defining the rank of a binary irrational to be ∞. We will denote the rank of x by $r(x)$. Applied to the binary sequences, the doubling function d is the *shift operator*:

Exercise 24. Show that $d(x)_i = x_{i+1}$ for $x \in [0,1)$

Bold play in red and black can be elegantly described by comparing the bits of the initial fortune with the random bits that describe the outcomes of the games .

Exercise 25. Suppose that gambler starts with initial fortune $x \in [0,1)$. Show that the gambler eventually reaches the target 1 if and only if there exists a positive integer k such that $I_j = 1 - x_j$ for $j \in \{1, 2, \ldots, k-1\}$ and $I_k = x_k$. That is, the gambler wins if and only if when the game bit agrees with the corresponding fortune bit for the first time, that bit is 1.

The random variable whose bits are the complements of the game bits will play an important role in our analysis. Thus, let

$$W = \sum_{i=1}^{\infty} \frac{1 - I_j}{2^j}$$

Note that W is a well defined random variable taking values in $[0, 1]$

Exercise 26. Suppose that the gambler starts with initial fortune $x \in [0, 1]$. Use the result of the Exercise 25 to show that the gambler reaches the target 1 if and only if $W < x$.

Exercise 27. Show that W has a continuous distribution. That is, show that $\mathbb{P}(W = x) = 0$ for any $x \in [0, 1]$

From the Exercises 26 and 27, it follows that F is simply the distribution function of W:

$$F(x) = \mathbb{P}(W \leq x), \quad x \in [0, 1]$$

In particular, F is an increasing function, and since W has a continuous distribution, F is a continuous function.

Exercise 28. Show that the success function F is the unique continuous solution of the functional equation in Exercise 23.

a Use mathematical induction on the rank to show that any two solutions of must agree at the binary rationals.

b Use part (a) and continuity to show that any two continuous solutions of the functional equation must agree for all x.

If we introduce a bit more notation, we can give nice expression for $F(x)$, and later for the expected number of games $G(x)$. Let $p_0 = p$ and $p_1 = q = 1-p$.

Exercise 29. Use Exercise 25 to show that

$$F(x) = \sum_{n=1}^{\infty} p_{x_1} \cdots p_{x_{n-1}} p x_n$$

No, the equation in Exercise 28 does not have a misprint. The last part really is p *times* x_n. Thus, only terms with $x_n = 1$ are included in the sum, Again, the player must win a game when her current fortune is in $[\frac{1}{2}, 1]$ (which of course happens with probability p and turns out to be the last game). Prior to this, she must win when her fortune is in $[0, \frac{1}{2})$ and lose when her fortune is in $(\frac{1}{2}, 1]$, so that the game does not end earlier.

Exercise 30. Show that F is strictly increasing on $[0, 1]$. This means that the distribution of W has support $[0, 1]$; that is, there are no subintervals of $[0, 1]$ that have positive length, but 0 probability.

Exercise 31. In particular, show that

a $F(\frac{1}{8}) = p^3$

b $F(\frac{2}{8}) = p^2$

c $F(\frac{3}{8}) = p^2 + p^2 q$

d $F(\frac{4}{8}) = p$

e $F(\frac{5}{8}) = p + p^2 q$

f $F(\frac{6}{8}) = p + pq$

g $F(\frac{7}{8}) = p + pq + pq^2$

Exercise 32. Suppose that $p = \frac{1}{2}$. Show that $F(x) = x$ for $x \in [0,1]$ in two ways:

a Using the functional equation in Exercise 23.

b Using the representation in Exercise 29.

Thus, for $p = \frac{1}{2}$ (fair trials), the probability that the bold gambler reaches the target fortune a starting from the initial fortune x is x/a, just as it is for the timid gambler. Note also that the random variable W has the uniform distribution on $[0,1]$. When $p \neq \frac{1}{2}$, the distribution of W is quite strange. To state the result succinctly, we will indicate the dependence of the of the probability measure $\mathbb{P}$ on the parameter $p \in (0,1)$. First we define

$$C_p = \left\{ x \in [0,1] : \frac{1}{n} \sum_{i=1}^{n} (1 - x_i) \to p \text{ as } n \to \infty \right\}$$

Thus, C_p is the set of $x \in [0,1]$ for which the relative frequency of 0's in the binary expansion is p. Of course, if $p \neq t$ then C_p and C_t are disjoint.

Exercise 33. Use the strong law of large numbers to show that

a $\mathbb{P}_p(W \in C_p) = 1$ for $p \in (0,1)$.

b $\mathbb{P}_p(W \in C_t) = 0$ for p, $t \in (0,1)$, with $p \neq t$.

Exercise 34. Show that when $p \neq \frac{1}{2}$, W does not have a probability density function, even though it has a continuous distribution. The following steps outline a proof by contradiction

a Suppose that W has probability density function f.

b Then $\mathbb{P}_p(W \in C_p) = \int_{C_p} f(x) dx$.

c By Exercise 33 $\mathbb{P}(W \in C_p) = 1$.

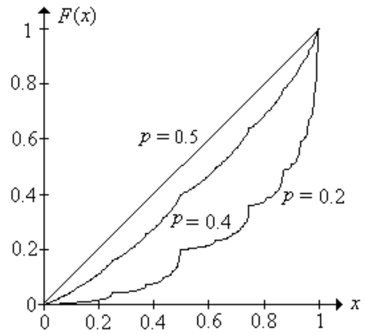

Figure 6: The graph of F for various values of p

d But also $\int_{C_p} 1dx = \mathbb{P}_{\frac{1}{2}}(W \in C_p) = 0$ That is, C_p has Lebesgue measure 0.

e Hence $\int_{C_p} f(x)dx = 0$

When $p \neq \frac{1}{2}$, F has derivative 0 at almost every point in $[0, 1]$ even though it is strictly increasing. Thus, W has a *singular continuous distribution*. Such distributions are usually considered exotic, so one of the great features of red and black is that it's an honest applied problem yet gives rise to such a distribution.

The Expected Number of Trials

Let $G(x) = \mathbb{E}(N|X_0 = x)$ for $x \in [0, 1]$, the expected number of trials starting at x. For any other target fortune $a \in (0, \infty)$, the expected number of trials starting at $x \in [0, a]$ is just $G(x/a)$.

Exercise 35. By conditioning on the outcome of the first game, show that G satisfies the functional equation in (a). Show directly that G satisfies the boundary conditions (b).

a $G(x) = \begin{cases} 1 + pG(2x), & x \in (0, \frac{1}{2}] \\ 1 + qG(2x - 1), & x \in [\frac{1}{2}, 1) \end{cases}$

b $G(0) = 0$, $G(1) = 0$

Note, interestingly, that the functional equation is not satisfied at $x = 0$ or $x = 1$. As before, we can give an alternate analysis using the binary representation of an initial fortune $x \in [0, 1]$.

Exercise 36. Suppose that the initial fortune of the gambler is $x \in [0, 1]$. Show that $N = \min\{k \in \mathbb{N}_+ : I_k = x_k \text{ or } k = r(x)\}$.

a If x is a binary rational then N takes values in the set $\{1, 2, \ldots, r(x)\}$. Play continues until the game number agrees with the rank of the fortune or a game bit agrees with the corresponding fortune bit, whichever is smaller. In the first case, the penultimate fortune is $\frac{1}{2}$, the only fortune for which the next game is always final.

b If x is a binary irrational then N takes values in $\mathbb{N}_+$. Play continues until a game bit agrees with a corresponding fortune bit.

We can give an explicit formula for the expected number of trials $G(x)$ in terms of the binary representation of x.

Exercise 37. Suppose that $x \in [0, 1]$ and recall our special notation: $p_0 = p$, $p_1 = q = 1 - p$. Show that

$$G(x) = \sum_{n=0}^{r(x)-1} p_{x_1} \cdots p_{x_n}$$

a Note that the $n = 0$ term is 1, since the product is empty.

b The sum has a finite number of terms if x is a binary rational.

c The sum has an infinite number of terms if x is a binary irrational.

Exercise 38. Use the result of the Exercise 37 to verify the following values:

a $G(\frac{1}{8}) = 1 + p + p^2$.

b $G(\frac{2}{8}) = 1 + p$.

c $G(\frac{3}{8}) = 1 + p + pq$,

d $G(\frac{4}{8}) = 1$.

e $G(\frac{5}{8}) = 1 + q + pq$

f $G(\frac{6}{8}) = 1 + q$

g $G(\frac{7}{8}) = 1 + q + q^2$

Exercise 39. Suppose that $p = \frac{1}{2}$. Use Exercise 37 to show that

$$G(x) = \begin{cases} 2 - \frac{1}{2^{r(x)-1}}, & x \text{ is a binary rational} \\ 2, & x \text{ is a binary irrational} \end{cases}$$

Exercise 40. For fixed x, show that G is continuous as a function of p.

However, as a function of the initial fortune x, for fixed p, the function G is very irregular.

Exercise 41. Show that G is discontinuous at the binary rationals in $[0, 1]$ and continuous at the binary irrationals.

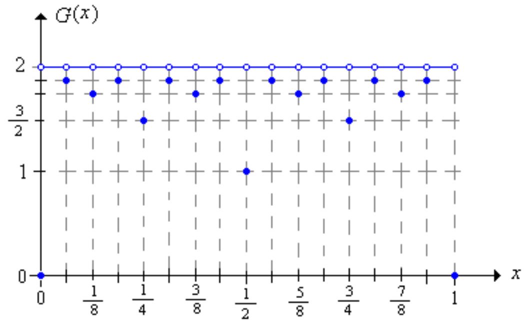

Figure 7: The expected number of games in bold play with fair games.

4 Optimal Strategies

Recall that the stopping rule for red and black is to continue playing until the gambler is ruined or her fortune reaches the target fortune a. Thus, the gambler's strategy is to decide how much to bet on each game before she must stop. Suppose that we have a class of strategies that correspond to certain valid fortunes and bets; A will denote the set of fortunes and B_x will denote the set of valid bets for $x \in A$. For example, sometimes (as with timid play) we might want to restrict the fortunes to set of integers $\{0, 1, \ldots, a\}$; other times (as with bold play) we might want to use the interval $[0, 1]$ as the fortune space. As for the bets, recall that the gambler cannot bet what she does not have and does not bet more than she needs in order to reach the target. Thus, a betting function S must satisfy

$$S(x) = \min\{x, a - x\}, \quad x \in A$$

Moreover, we always restrict our strategies to those for which the stopping time is N finite.

The *success function* of a strategy is the probability that the gambler reaches the target a with that strategy, as a function of the initial fortune x. A strategy with success function V is optimal if for any other strategy with success function U, we have $U(x) \leq V(x)$ for $x \in A$.

Exercise 42. Show if there exists an optimal strategy, then the optimal success function is unique.

However, there may not exist an optimal strategy or there may be several optimal strategies. Moreover, the optimality question depends on the value of the game win probability p, in addition to the structure of fortunes and bets.

A Condition for Optimality

Here is our main theorem.

Theorem 1. A strategy S with success function V is optimal if

$$pV(x+y) + qV(x-y) \le V(x), \quad x \in A,\, y \in B_x$$

The proof is sketched in the following exercises. First, suppose we start with a given strategy S that has success function V. We modify S as follows. if the initial fortune is $x \in A$, we pick $y \in B_x$ and then bet y on the first game; thereafter we follow strategy S. Let's call the new strategy T and its success function U.

Exercise 43. Condition on the outcome of the first game to show that

$$U(x) = pV(x+y) + qV(x-y)$$

Thus, the Theorem 1 can be restated as follows: If S is optimal with respect to the class of strategies in Exercise 43, then S is optimal over all strategies.

Now, with S and V as before, let T be an arbitrary strategy with success function U. As usual, let $(X_0, X_1, X_2, \ldots)$ denote the sequence of fortunes, $(Y_1, Y_2, \ldots)$ the sequence of bets, and N the stopping time, all under strategy T. Note that the random variable $V(X_N)$ can be interpreted as the probability of winning if the gambler's strategy is replaced by strategy S after time n.

Exercise 44. Condition on the outcome of game n to show that

$$\mathbb{E}[V(X_N)|X_0 = x] = \mathbb{E}[pV(X_{n-1} + Y_n) + qV(X_{n-1} - Y_n)|X_0 = x]$$

We can now finish the proof of Theorem 4.1.

Exercise 45. Suppose that the success function V under strategy S satisfies the condition in Theorem 1.

a Use Exercise 44 and the optimality condition to show

$$\mathbb{E}[V(X_n)|X_0 = x] \le \mathbb{E}[V(X_{n-1}|X_0 = x], \quad n \in \mathbb{N}_+,\, x \in A.$$

b Use (a) to show that $E[V(X_n)|X_0 = x] \le V(x)$ for $n \in \mathbb{N}_+$ and $x \in A$.

c Let $n \to \infty$ in (b) to show that $\mathbb{E}[V(X_N)|X_0 = x] \le V(x)$ for $x \in A$.

d Show that $\mathbb{E}[V(X_N)|X_0 = x] = U(x)$ for $x \in A$

e Conclude that $U(x) \le V(x)$ for $x \in A$.

Favorable Trials with a Minimum Bet

Suppose now that $p \ge \frac{1}{2}$ so that the trials are favorable (or at least not unfair) to the gambler. Next, suppose that all bets must be multiples of a basic unit, which we might as well assume is \$1. Of course, real gambling houses have this restriction. Thus the set of valid fortunes is $A = \{0, 1, \ldots, a\}$ and the set of valid bets for $x \in A$ is $B_x = \{0, 1, \ldots, \min\{x, a-x\}\}$. Our main result for this subsection is

Theorem 2. Timid play is an optimal strategy.

The proof will be constructed in the following two exercises. First, recall the success function f for timid play, derived in Exercises 7 and 8, and the optimality condition in Theorem 1.

Exercise 46. Show first that optimality condition holds if $p = \frac{1}{2}$

Exercise 47. Show that optimality condition holds if $p > \frac{1}{2}$. Here are the steps.

a Show that the condition for optimality is equivalent to

$$p(q/p)^{x+y} + q(q/p)^{x-y} \geq (q/p)^x$$

b Show that the inequality in (a) is equivalent to

$$pq(p^y - q^y)(p^{y-1} - q^{y-1}) \leq 0$$

c Show that the inequality in (b) holds since $p > \frac{1}{2}$.

Favorable Trials without a Minimum Bet

We will now assume that the house allows arbitrarily small bets and that $p > \frac{1}{2}$, so that the trials are strictly favorable. In this case it is natural to take the target as the monetary unit so that the set of fortunes is $A = [0, 1]$, and the set of bets for $x \in A$ is $B_x = [0, \min\{x, 1 - x\}]$. Our main result, and a sketch of the proof, are given in the following exercise. The results for timid play will play an important role in the analysis, so we will let $f(j, a)$ denote the probability of reaching an integer target a, starting at the integer $j \in [0, a]$ with unit bets.

Exercise 48. Show that the optimal success function is $V(x) = 1$ for $x \in [0, 1]$

a Fix a rational initial fortune $x = k/n \in [0, 1]$. Let m be a positive integer and suppose that, starting at x, the gambler bets $\frac{1}{mn}$ on each game.

b Show the strategy in (a) is equivalent to timid play with target fortune mn and initial fortune mk.

c The probability of reaching the target 1 under the strategy in (b) is $f(mk, mn)$.

d Show that $f(mk, mn) \to 1$ as $m \to \infty$.

e From (d), show that $V(x) = 1$ if $x \in [0, 1]$ is rational.

f From (e) and the fact that V is increasing to show that $V(x) = 1$ for all $x \in [0, 1]$

Unfair Trials

We will now assume that $p \leq \frac{1}{2}$ so that the trials are unfair, or at least not favorable. As before, we will take the target fortune as the basic monetary unit and allow any valid fraction of this unit as a bet. Thus, the set of fortunes is $A = [0, 1]$ and the set of bets for $x \in A$ is $B_x = [0, \min\{x, 1 - x\}]$. Our main result for this section is.

Theorem 3. Bold play is an optimal strategy.

As usual, the proof will be constructed through a series of exercises. First, recall the success function F for bold play, and the optimality condition in Theorem 1. Let $D(x, y) = F(\frac{x+y}{2}) - pF(x) + qF(y)$.

Exercise 49. Show that the optimality condition is equivalent to $D(x, y) \leq 0$ for $0 \leq x \leq y \leq 1$.

Exercise 50. Use the continuity of F to show that it suffices to prove the inequality in Exercise 49 when x and y are binary rationals.

The proof is completed by using induction on the rank of x and y.

Exercise 51. Show that the inequality in Exercise 49 holds when x and y have rank 0:

a $x = 0$, $y = 0$

b $x = 0$, $y = 1$

c $x = 1$, $y = 1$

Exercise 52. Suppose that the inequality in Exercise 49 holds when x and y have rank m or less. Show that the inequality holds when x and y have rank $m + 1$ or less.

a Suppose that $x \leq y \leq \frac{1}{2}$ Show that $D(x, y) = pD(2x, 2y)$.

b Suppose that $\frac{1}{2} \leq x \leq y$. Show that $D(x, y) = qD(2x - 1, 2y - 1)$.

c Suppose that $x \leq (x + y)/2 \leq \frac{1}{2} \leq y$ and that $2y - 1 \leq 2x$. Show that $D(x, y) = (q - p)F(2y - 1) + qD(2y - 1, 2x)$.

d Suppose that $x \leq (x + y)/2 \leq \frac{1}{2} \leq y$ and that $2x \leq 2y - 1$. Show that $D(x, y) = (q - p)F(2x) + qD(2x, 2y - 1)$.

e Suppose that $x \leq \frac{1}{2} \leq (x + y)/2 \leq y$ and that $2y - 1 \leq 2x$. Show that $D(x, y) = p(q - p)[1 - F(2x)] + pD(2y - 1, 2x)$.

f Suppose that $x \leq \frac{1}{2} \leq (x + y)/2 \leq y$ and that $2x \leq 2y - 1$. Show that $D(x, y) = p(q - p)[1 - F(2y - 1)] + pD(2x, 2y - 1)$.

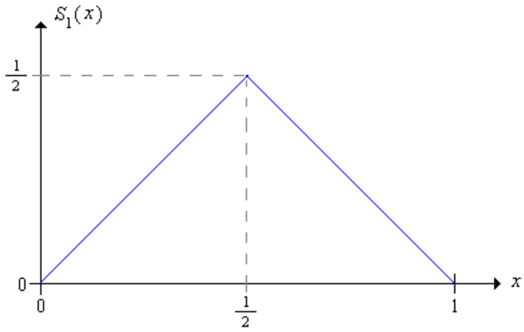

Figure 8: The betting function for bold play.

Other Optimal Strategies in the Sub-Fair Case

Consider again the sub-fair case where $p \leq \frac{1}{2}$ so that the trials are not favorable to the gambler. Bold play is not the only optimal strategy; amazingly, there are infinitely many optimal strategies. Recall first that the bold strategy has betting function

$$S_1(x) = \min\{x, 1-x\} = \begin{cases} x, & x \in [0, \frac{1}{2}] \\ 1-x, & x \in [\frac{1}{2}, 1] \end{cases}$$

Now consider the following strategy, which we will refer to as the *second order bold strategy*:

- With fortune $x \in (0, \frac{1}{2})$, play boldly with the object of reaching $\frac{1}{2}$ before falling to 0.
- With fortune $x \in (\frac{1}{2}, 1)$ play boldly with the object of reaching 1 without falling below $\frac{1}{2}$.
- With fortune $\frac{1}{2}$, bet $\frac{1}{2}$.

Exercise 53. Show that the second order bold strategy has betting function S_2 given by

$$S_2(x) = \begin{cases} x, & x \in [0, \frac{1}{4}] \\ \frac{1}{2} - x, & x \in [\frac{1}{4}, \frac{1}{2}) \\ \frac{1}{2}, & x = \frac{1}{2} \\ x - \frac{1}{2}, & x \in (\frac{1}{2}, \frac{3}{4}] \\ 1 - x, & x \in [\frac{3}{4}, 1] \end{cases}$$

Let F_2 denote the success function associated with strategy S_2. Our main theorem for this section is

Theorem 4. The second order bold strategy is optimal. That is, $F_2 = F$

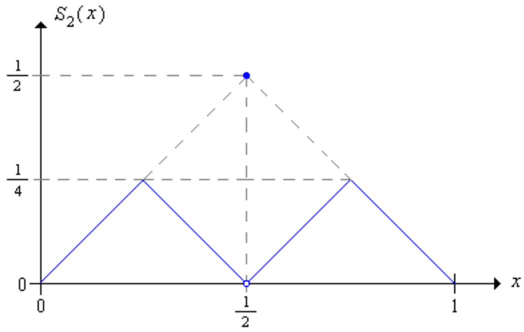

Figure 9: The betting function for the second order bold strategy.

Proof. The main tool is the functional equation for the success function F under bold play. First, suppose that the player starts with fortune $x \in (0, \frac{1}{2})$ and uses strategy S_2. Note that the player reaches the target 1 if an only if she reaches $\frac{1}{2}$ and then wins the final game. Thus, consider the sequence of fortunes until the player reaches 0 or $\frac{1}{2}$. If we double the fortunes, then we have the fortune sequence under the ordinary bold strategy S, starting at $2x$ and terminating at either 0 or 1. It follows that

$$F_2(x) = pF(2x) = F(x)$$

Suppose now that the player starts with fortune $x \in (\frac{1}{2}, 1)$. and uses strategy S_2. Note that the player reaches the target 1 if an only if she reaches 1 without falling back to $\frac{1}{2}$, or if she falls back to $\frac{1}{2}$ and then wins the final game. Thus, consider the sequence of fortunes until the player reaches $\frac{1}{2}$ or 1. If we double the fortunes and subtract 1, then we have the fortune sequence under the ordinary bold strategy, starting at $2x - 1$ and terminating at either 0 or 1. It follows that

$$F_2(x) = F(2x - 1) + [1 - F(2x - 1)]p = p + qF(2x - 1) = F(x)$$

Of course, trivially $F_2(0) = F(0) = 0$, $F_2(1) = F(1) = 1$, and $F_2(\frac{1}{2}) = F(\frac{1}{2}) = p$ Thus,$F_2(x) = F(x)$ for all $x \in [0, 1]$. ☐

Once we understand how this construction is done, it's straightforward to define the third order bold strategy and show that it's optimal as well. The graph of the betting function is shown in Figure 10.

Exercise 54. Explicitly give the third order betting function and show that the strategy is optimal.

More generally, we can define the nth order bold strategy and show that it is optimal as well.

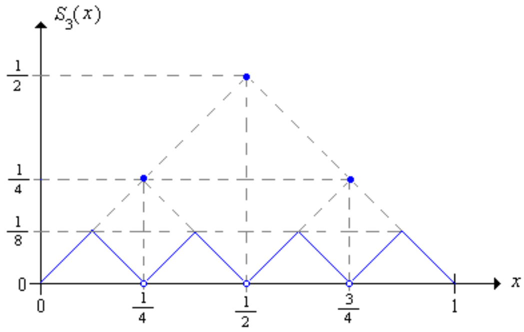

Figure 10: The betting function for the third order bold strategy.

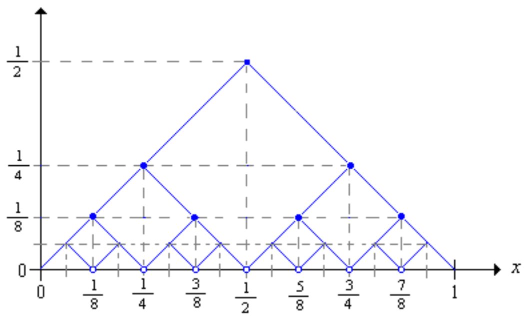

Figure 11: Betting functions for the first four bold strategies.

Exercise 55. Show that the sequence of bold strategies can be defined recursively from the basic bold strategy S_1 as follows:

$$S_{n+1}(x) = \begin{cases} \frac{1}{2} S_n(2x), & x \in [0, \frac{1}{2}) \\ \frac{1}{2}, & x = \frac{1}{2} \\ \frac{1}{2} S_n(2x-1), & x \in (\frac{1}{2}, 1] \end{cases}$$

Even more generally, we can define an optimal strategy T in the following way: for each $x \in [0, \frac{1}{2}]$, select $n_x \in \mathbb{N}_+$ and let $T(x) = S_{n_x}(x)$. The graph in Figure 11 shows a few of the graphs of the bold strategies. For an optimal strategy T, we just need to select, for each x a bet on one of the graphs.

5 Summary

In the game of Red and Black, a gambler plays Bernoulli trials (independent, probabilistically identical games with win probability p) until she is either ruined or reaches a fixed target a. Our main interest is her success probability, the probability of reaching her target. A secondary interest is the expected number of games played. Mostly we are concerned with the unfair case $p < \frac{1}{2}$ since this is the unfortunate situation at real casinos. We have compared two very different strategies:

With timid play, the gambler makes a small constant bet on each game until she is ruined or reaches the target. This turns out to be a very bad strategy in unfair games, but does have the advantage of a relatively large expected number of games. If you play this strategy, you will almost certainly be ruined, but at least you get to gamble for a while.

With bold play, the gambler bets her entire fortune or what she needs to reach the target, whichever is smaller. This is an optimal strategy in the unfair case; no strategy can do better. But it's very quick! If you play this strategy, there's a good chance that your gambling will be over after just a few games (often just one!).

Amazingly, bold play is not uniquely optimal in the unfair case. Bold play can be rescaled to produce an infinite sequence of optimal strategies. These higher order bold strategies can have large expected numbers of games, depending on the initial fortune (and assuming that the casino is nice enough to allow you to make the weird fractional bets that will be required for the strategy).

However the point of this article is not to make you a better gambler (in spite of the title), but to uncover some beautiful mathematics. The study of bold play, and its higher-order variants, leads to an interesting mix of probability, dynamical systems, and the binary representation of numbers in $[0, 1]$ (for both the fortunes and the probabilities). The study of bold play leads to singular continuous probability distributions, that is, continuous distributions that do not have density functions. Generalizations of red and black and bold play are interesting research topics in mathematics to this day; for a relatively recent article, see [3]. If you want to learn more about the mathematics of gambling, a good place to start are the books [1] (the inspiration for this article) and [2].

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