3.2 Roulette and Markov Chains

In this section we will be discussing an application of systems of recursion equations called Markov Chains. Markov Chains are used in numerous applications to model different scenarios. Before we formally define Markov Chains, we will first introduce the concept behind them using a well known game called Roulette.

3.2.1 Roulette – A Question of Strategy

Figure 3.5: A roulette wheel

Figure 3.5 shows a roulette wheel. It has a total of 38 slots – 18 are red, 18 are black, and two are green. A player often bets on either black or red. These bets are "even money" bets. The player places the amount he wishes to bet on the table. If the ball lands in a slot whose color matches the color on which the player bet, then the croupier puts an amount equal to the player's bet on top of his bet and pushes the pile to the player. If the ball lands in a slot of a different color, then the croupier rakes in the player's bet. For example, if the player starts with \$30.00 and bets \$10.00 on one spin of the wheel on black or red then after that spin he will have either \$40.00 if he wins or \$20.00 if he loses. The player's chances of winning on each spin of the wheel are 18/38 because 18 of the slots match the color on which the bet was placed.

Suppose that a player has \$30.00 and wants to win an additional \$30.00 to give himself a total of \$60.00. We want to examine and contrast two of his many possible strategies.

- **The aggressive strategy:** The player strides confidently up to the table and places a single bet of \$30.00 on the first spin of the wheel. He either wins or loses. If he loses he smiles bravely and leaves. If he wins he smiles triumphantly, pockets his \$60.00, and leaves. With this strategy his chances of winning are 18/38 or 47.37%.
- **The conservative strategy:** The player walks hesitantly up to the table and places a bet of \$10.00 on the first spin of the wheel. Whatever happens, he places another bet of \$10.00 on the next spin of the wheel. He continues in this way, betting \$10.00 on each spin of the wheel, until he either reaches his goal of \$60.00 or he goes broke.

This is an example of a common kind of choice that people often face. For example, investors must often decide whether to place all their money in a single investment or to diversify their holdings, placing smaller amounts in each of several investments.

Problem 3.2.1 *Do you think the player is more likely to win using the aggressive strategy, using the conservative strategy, or that it makes no difference which strategy is used?*

We already know that the probability that the player wins with the aggressive strategy is 18/38, or roughly 47%. Thus, our real problem is determining the probability that the player wins with the conservative strategy.

One way to attack this problem is by experimentation. A player could try the conservative strategy many times and keep track of how often he wins. There are seven possible situations or **states** in which he might find himself – according to how much money he has.

- **State 1:** \$0.00. He is broke and has lost the game. He is no longer playing.
- **State 2:** \$10.00. He is still playing.
- **State 3:** \$20.00. He is still playing.
- **State 4:** \$30.00. He is still playing.
- **State 5:** \$40.00. He is still playing.
- **State 6:** \$50.00. He is still playing.
- **State 7:** \$60.00. He has reached his goal and has won. He is no longer playing.

This situation involves uncertainty and probability – we cannot determine whether the player will win or lose with the conservative strategy but we can determine how likely he is to win or lose. Similarly, we cannot determine how much money the player will have after two spins of the roulette wheel (that is, what state he is in) but we can determine how likely he is to be in each state. We use a number, p, between zero and one to describe the likelihood or **probability** of an event. If $p = 1$ then the event is certain – that is, it has already occurred or it is certain that it will occur. If $p = 0$ then the event either did not occur or it is certain that it will not occur. If, for example, $p = 0.25$, then the event will occur 1/4 of the time. For example, if you flip a fair (or balanced) coin then the probability of heads is 1/2, or if you roll a six-sided die then the probability of coming up with a four is 1/6.

In our situation, we need to keep track of seven probabilities – the probability of each of the seven states. For this purpose, we use seven-dimensional vectors, called **probability vectors**,

$$\vec{p} = \langle p_1,\ p_2,\ p_3,\ p_4,\ p_5,\ p_6,\ p_7\rangle.$$

Each of the seven entries in this vector is a number between zero and one indicating the probability of the corresponding state. Because there are only seven possible states,

$$p_1 + p_2 + p_3 + p_4 + p_5 + p_6 + p_7 = 1.$$

When the player starts playing he has \$30.00 and is in state 4. Thus, the initial or starting probability vector is

$$\vec{p}_0 = \langle 0,\ 0,\ 0,\ 1,\ 0,\ 0,\ 0\rangle.$$

Notice the subscript 0 in the notation $\vec{p}_0$. This subscript indicates that we are talking about the probability after zero spins of the wheel. We will use the notation $\vec{p}_1$ for the probability vector after one spin of the wheel; the notation $\vec{p}_2$ for the probability vector after two spins of the wheel; and $\vec{p}_n$ for the probability vector after n spins of the wheel. Don't confuse the notation p_1 and $\vec{p}_1$. In this discussion the notation $\vec{p}_1$ refers to the specific vector of probabilities after one spin of the wheel. We used the notation p_1 above to denote the first element of a general probability vector, $\vec{p}$.

Additionally, we will use generic notation to describe any probability vector, $\vec{p}_k$, and its entries, $p_{i,k}$. This describes the probability of being in state i after k spins of the wheel.

$$\vec{p}_k = \langle p_{1,k},\ p_{2,k},\ p_{3,k},\ p_{4,k},\ p_{5,k},\ p_{6,k},\ p_{7,k}\rangle.$$

Figure 3.6 on page 241 is the key to analyzing the conservative strategy. It describes how a player moves from one state to another on each spin of the wheel. It is called a **transition diagram**.

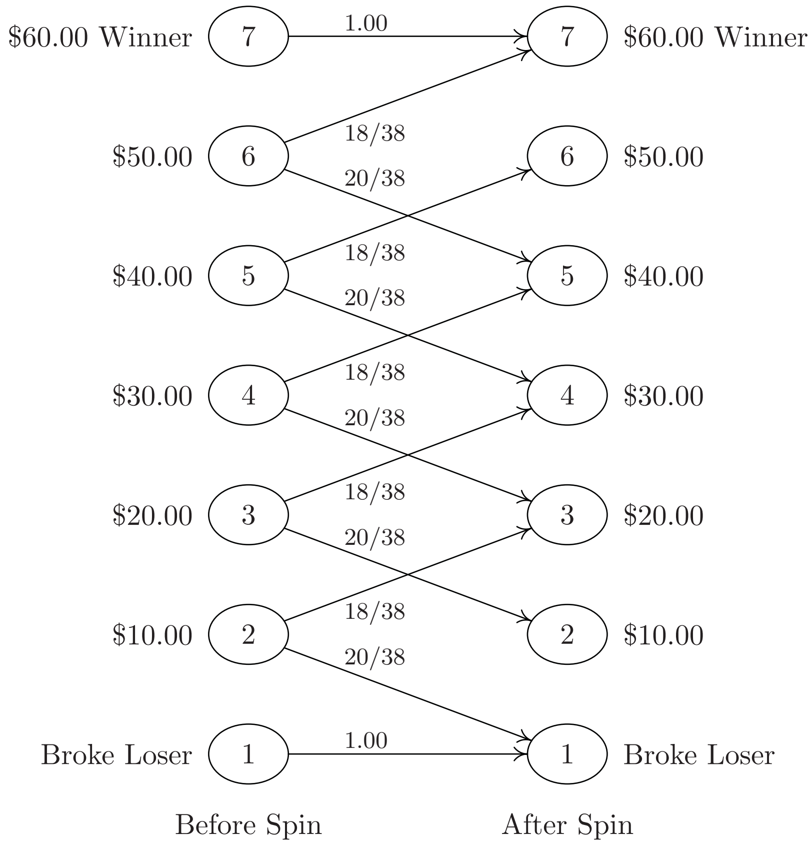

Figure 3.6: Transition diagram for one spin of the wheel

The circles on the left side of this transition diagram show the states that the player might be in before the spin and the circles on the right side show the states that the player might be in after the spin. The arrows indicate the possible changes and their probabilities.

As one example, if a player is in state 1 then he is broke and no longer playing, so he will remain in that state. Notice there is only one arrow leading from state 1 and that arrow goes to the same state, state 1, and has probability 1. As another example, suppose that a player is in state 3 and has $20.00. Then he will either win or lose $10.00 when the wheel is spun. Thus, after the spin he will be in either state 2 ($10.00) or state 4 ($30.00). There are two arrows leading from state 3 on the left and they go to states 2 and 4. Because the probability of winning on one spin of the wheel is 18/38, the probability on the arrow going from state 3 to state 4 is 18/38. Because the probability of losing on each spin is 20/38, the probability on the arrow going from state 3 to state 2 is 20/38. This same information is shown in Table 3.3. This table is called the **transition table**.

	from state 1	from state 2	from state 3	from state 4	from state 5	from state 6	from state 7
to state 1	1	20/38	0	0	0	0	0
to state 2	0	0	20/38	0	0	0	0
to state 3	0	18/38	0	20/38	0	0	0
to state 4	0	0	18/38	0	20/38	0	0
to state 5	0	0	0	18/38	0	20/38	0
to state 6	0	0	0	0	18/38	0	0
to state 7	0	0	0	0	0	18/38	1

Table 3.3: The transition table for the conservative strategy

We can capture the same information in the **transition matrix**

$$T = \begin{bmatrix} 1 & 20/38 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 20/38 & 0 & 0 & 0 & 0 \\ 0 & 18/38 & 0 & 20/38 & 0 & 0 & 0 \\ 0 & 0 & 18/38 & 0 & 20/38 & 0 & 0 \\ 0 & 0 & 0 & 18/38 & 0 & 20/38 & 0 \\ 0 & 0 & 0 & 0 & 18/38 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 18/38 & 1 \end{bmatrix}$$

Vectors and matrix multiplication were designed to handle these types of problems. Let

$$\vec{p}_{n-1} = \langle p_{1,n-1},\ p_{2,n-1},\ p_{3,n-1},\ p_{4,n-1},\ p_{5,n-1},\ p_{6,n-1},\ p_{7,n-1}\rangle$$

denote the probability vector of being in any state after $(n-1)$ spins of the wheel and let

$$\vec{p}_n = \langle p_{1,n},\ p_{2,n},\ p_{3,n},\ p_{4,n},\ p_{5,n},\ p_{6,n},\ p_{7,n}\rangle$$

denote the probability vector of being in any state after n spins of the wheel. Notice that

$$\begin{aligned}
p_{1,n} &= p_{1,n-1} + (20/38)p_{2,n-1} \\
p_{2,n} &= (20/38)p_{3,n-1} \\
p_{3,n} &= (18/38)p_{2,n-1} + (20/38)p_{4,n-1} \\
p_{4,n} &= (18/38)p_{3,n-1} + (20/38)p_{5,n-1} \\
p_{5,n} &= (18/38)p_{4,n-1} + (20/38)p_{6,n-1} \\
p_{6,n} &= (18/38)p_{5,n-1} \\
p_{7,n} &= (18/38)p_{6,n-1} + p_{7,n-1}.
\end{aligned}$$

Using these recursion equations, we can compute successive probability vectors.

Example 3.2.1 *Find the probability vector $\vec{p}_1$ that describes what we can expect after one spin of the wheel.*

To begin the solution for this example, recall the initial probability vector

$$\vec{p}_0 = \langle 0,\ 0,\ 0,\ 1,\ 0,\ 0,\ 0\rangle.$$

We can now calculate the entries of $\vec{p}_1$, the probability vector after 1 spin of the wheel, using the equations above.

$$\begin{aligned}
p_{1,1} &= p_{1,0} + (20/38)p_{2,0} = (0) + (20/38)(0) = 0\\
p_{2,1} &= (20/38)p_{3,0} = (20/38)(0) = 0\\
p_{3,1} &= (18/38)p_{2,0} + (20/38)p_{4,0} = (18/38)(0) + (20/38)(1) = (20/38)\\
p_{4,1} &= (18/38)p_{3,0} + (20/38)p_{5,0} = (18/38)(0) + (20/38)(0) = 0\\
p_{5,1} &= (18/38)p_{4,0} + (20/38)p_{6,0} = (18/38)(1) + (20/38)(0) = (18/38)\\
p_{6,1} &= (18/38)p_{5,0} = (18/38)(0) = 0\\
p_{7,1} &= (18/38)p_{6,0} + p_{7,0} = (18/38)(0) + (0) = 0
\end{aligned}$$

Therefore, $\vec{p}_1 = \langle 0,\ 0,\ (20/38),\ 0,\ (18/38),\ 0,\ 0\rangle$. This accurately describes what we already know. After one spin of the wheel, there is a $\frac{20}{38}\%$ chance of losing \$10 and ending up in state 3 with \$20. Similarly, there is a $\frac{18}{38}\%$ chance of winning \$10 and ending up in state 5 with \$40.

Problem 3.2.2 *Find the probability vector $\vec{p}_2$ that describes what we can expect after two spins of the wheel.*

Problem 3.2.3 *Find the probability vector $\vec{p}_3$ that describes what we can expect after three spins of the wheel.*

As with single recursion equation, however, repeating this process for a large number of iterations is time consuming and cumbersome. Notice however, that these results are exactly what we get if we perform the matrix multiplication

$$\vec{p}_n = T\vec{p}_{n-1}$$

where we think of the vectors $\vec{p}_n$ and $\vec{p}_{n-1}$ as column-vectors. The initial condition is

$$\vec{p}_0 = \langle 0,\ 0,\ 0,\ 1,\ 0,\ 0,\ 0\rangle, \text{ or } \qquad \vec{p}_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Using the matrix-vector form to calculate $\vec{p}_1$ yields

$$\begin{aligned}
\vec{p}_1 &= T\vec{p}_0 \\
&= \begin{bmatrix} 1 & 20/38 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 20/38 & 0 & 0 & 0 & 0 \\ 0 & 18/38 & 0 & 20/38 & 0 & 0 & 0 \\ 0 & 0 & 18/38 & 0 & 20/38 & 0 & 0 \\ 0 & 0 & 0 & 18/38 & 0 & 20/38 & 0 \\ 0 & 0 & 0 & 0 & 18/38 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 18/38 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \\
\vec{p}_1 &= \begin{bmatrix} 0 \\ 0 \\ \frac{20}{38} \\ 0 \\ \frac{18}{38} \\ 0 \\ 0 \end{bmatrix}.
\end{aligned}$$

Problem 3.2.4 *Compute $\vec{p}_2$ using matrix multiplication as described above. Compare your answer with your answer to Problem 3.2.2.*

Problem 3.2.5 *Compute $\vec{p}_3$ using matrix multiplication as described above. Compare your answer with your answer to Problem 3.2.3.*

Problem 3.2.6 *Find the probability vector* $\vec{p}_{20}$ *that describes what we can expect after 20 spins of the wheel playing the conservative strategy. What is the probability that a player playing the conservative strategy will have won after twenty spins of the wheel? What is the probability that a player playing the conservative strategy will have lost after twenty spins of the wheel? What is the probability that a player playing the conservative strategy will still be playing after twenty spins of the wheel?*

Problem 3.2.7 *Compare the conservative and aggressive strategies.*

Problem 3.2.8 *Another player has $50.00. She plans to bet $50.00 on either red or black on each spin of the wheel until she either goes broke or reaches $150.00. What is her probability of winning? Before answering this question using the techniques above, make a rough guess. When you are done, compare your rough guess with your answer.*

Problem 3.2.9 *Another player has $50.00. She plans to bet $50.00 on either red or black on each spin of the wheel until she either goes broke or reaches $200.00. What is her probability of winning? Before answering this question using the techniques above, make a rough guess. When you are done, compare your rough guess with your answer.*

Problem 3.2.10 *How would the outcomes change if the two green slots were removed?*

3.2.2 Markov Chains

Our analysis of the roulette question was based on a powerful mathematical idea called **Markov Chains**. The following definition summarizes the elements of a Markov Chain.

Definition 3.2.1 *A* **Markov Chain** *consists of the following.*

- *A set S of n* **states**. *In the roulette problem the set S had seven states.*
- *An $n \times n$* **transition matrix**

$$T = \begin{bmatrix} t_{11} & t_{12} & \cdots & t_{1n} \\ t_{21} & t_{22} & \cdots & t_{2n} \\ \vdots & \vdots & & \vdots \\ t_{n1} & t_{n2} & \cdots & t_{nn} \end{bmatrix}$$

The element t_{ij} gives the probability of moving from state j to state i on each turn or play.

- *An* **initial probability vector**

$$\vec{p}_0 = \langle p_1,\ p_2, \ldots,\ p_n \rangle.$$

Using Definition 3.2.1, a sequence of probability vectors $\vec{p}_0,\ \vec{p}_1,\ \vec{p}_2, \ldots,\ \vec{p}_n, \ldots$, that describe the probabilities after n turns, can be computed using the equation

$$\vec{p}_n = T\vec{p}_{n-1}.$$

The following observations are important.

- Because the vectors $\vec{p}_n$ are all probability vectors, their entries are all between zero and one and the sum of their entries is always one.
- Because each column of the transition matrix T gives the probability of moving on one turn from one state to each of the other states, the entries in T are all between zero and one and the sum of the entries in each column is one.

Notice that a Markov Chain is also a *homogeneous linear discrete system* with as many variables as there are states.

Example 3.2.2 *Figure 3.7 shows a diagram of a four-celled organism. A chemical is injected into the leftmost cell – the cell labeled "Cell 1." The molecules of this chemical move randomly between adjacent cells. Every second the probability that a given molecule moves from one cell to each of the adjacent cells is* $1/5$. *Determine the distribution of the chemical in this organism after ten seconds. Determine the distribution of the chemical in this organism after 100 seconds.*

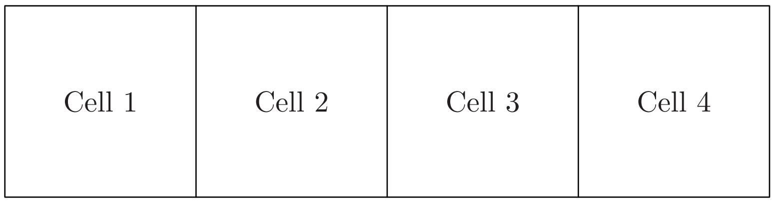

Figure 3.7: A four-celled organism

We can set this up as a Markov Chain problem. There are four states, corresponding to Cells 1, 2, 3, and 4. The initial probability vector is $\vec{p}_0 = \langle 1, 0, 0, 0 \rangle$ because the chemical is injected initially into Cell 1. The transition matrix is

$$T = \begin{bmatrix} 0.80 & 0.20 & 0 & 0 \\ 0.20 & 0.60 & 0.20 & 0 \\ 0 & 0.20 & 0.60 & 0.20 \\ 0 & 0 & 0.20 & 0.80 \end{bmatrix}.$$

Figure 3.8 on page 248 shows *Mathematica* code used to study this problem. After ten seconds, $\vec{p}_{10}$, $\approx 37.4\%$ of the chemical is in Cell 1; $\approx 29.9\%$ is in Cell 2; $\approx 19.8\%$ is in Cell 3 and $\approx 12.9\%$ is in Cell 4. After 100 seconds, p_{100}, $\approx 25\%$ of the chemical is in each cell.

3.2.3 Analytic Solutions of Homogeneous Linear Discrete Dynamical Systems with Many Variables

We have seen several examples of homogeneous linear discrete dynamical systems with many variables. If we use the notation

$$\vec{p}_n = A\vec{p}_{n-1}, \quad \vec{p}_0 = \vec{c}$$

this resembles (and is) a geometric sequence that uses vectors and matrices. If we iterate this DDS we hope to see a similar pattern emerge that we saw with single homogeneous recursion equations. Notice that

```
In[11]:= Clear[p]

         p[0] = (1
                 0
                 0
                 0);

         T = (0.80 0.20  0    0
              0.20  .60 0.20  0
               0   0.20 0.60 0.20
               0    0   0.20 0.80);

         p[n_] := p[n] = T.p[n - 1]

         MatrixForm[p[10]]
Out[15]//MatrixForm=
         (0.374266
          0.299333
          0.197644
          0.128757)

In[16]:= MatrixForm[p[100]]
Out[16]//MatrixForm=
         (0.250002
          0.250001
          0.249999
          0.249998)
```

Figure 3.8: Using *Mathematica* to study a four-celled organism

$$
\begin{aligned}
\vec{p}_1 &= A\vec{p}_0 = A\vec{c} \\
\vec{p}_2 &= A\vec{p}_1 = A(A\vec{c}) = A^2\vec{c} \\
\vec{p}_3 &= A\vec{p}_2 = A(A^2\vec{c})) = A^3\vec{c} \\
&\vdots \\
\vec{p}_n &= A^n\vec{c} = A^n\vec{p}_0.
\end{aligned}
$$

This pattern is what we hoped to see. This result yields the analytic solution

$$\vec{p}_n = A^n\vec{p}_0.$$

Note that

$$A^n = \underbrace{A \cdot A \cdots A}_{n-\text{times}}.$$

Later in this chapter we will develop simpler analytic solutions that make it easier to compute equilibrium vectors and understand the long–term behavior. For now, however, note that one way to gain some understanding of long–term behavior is by looking at the matrices A^n for very large values of n. Although this involves lots of matrix multiplication, the application of computer power can make the calculations easy. Consider the transition matrix from our analysis of the conservative roulette strategy.

$$T = \begin{bmatrix} 1 & 20/38 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 20/38 & 0 & 0 & 0 & 0 \\ 0 & 18/38 & 0 & 20/38 & 0 & 0 & 0 \\ 0 & 0 & 18/38 & 0 & 20/38 & 0 & 0 \\ 0 & 0 & 0 & 18/38 & 0 & 20/38 & 0 \\ 0 & 0 & 0 & 0 & 18/38 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 18/38 & 1 \end{bmatrix}$$

We can compute

$$T^{1024} = \begin{bmatrix} 1.000000 & 0.873977 & 0.733952 & 0.578369 & 0.405499 & 0.213420 & 0.000000 \\ 0.000000 & 0.000000 & 0.000000 & 0.000000 & 0.000000 & 0.000000 & 0.000000 \\ 0.000000 & 0.000000 & 0.000000 & 0.000000 & 0.000000 & 0.000000 & 0.000000 \\ 0.000000 & 0.000000 & 0.000000 & 0.000000 & 0.000000 & 0.000000 & 0.000000 \\ 0.000000 & 0.000000 & 0.000000 & 0.000000 & 0.000000 & 0.000000 & 0.000000 \\ 0.000000 & 0.000000 & 0.000000 & 0.000000 & 0.000000 & 0.000000 & 0.000000 \\ 0.000000 & 0.126023 & 0.266048 & 0.421631 & 0.594501 & 0.786580 & 1.000000 \end{bmatrix}.$$

This matrix can be used to determine what happens after 1024 spins of the wheel, since

$$\vec{p}_{1024} = T^{1024}\vec{p}_0.$$

Recall that our gambler started with \$30.00 and was in state 4 with probability 1. Thus

$$\vec{p}_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

and after 1024 spins

$$
\begin{aligned}
\vec{p}_{1024} &= \begin{bmatrix}
1.000000 & 0.873977 & 0.733952 & 0.578369 & 0.405499 & 0.213420 & 0.000000 \\
0.000000 & 0.000000 & 0.000000 & 0.000000 & 0.000000 & 0.000000 & 0.000000 \\
0.000000 & 0.000000 & 0.000000 & 0.000000 & 0.000000 & 0.000000 & 0.000000 \\
0.000000 & 0.000000 & 0.000000 & 0.000000 & 0.000000 & 0.000000 & 0.000000 \\
0.000000 & 0.000000 & 0.000000 & 0.000000 & 0.000000 & 0.000000 & 0.000000 \\
0.000000 & 0.000000 & 0.000000 & 0.000000 & 0.000000 & 0.000000 & 0.000000 \\
0.000000 & 0.126023 & 0.266048 & 0.421631 & 0.594501 & 0.786580 & 1.000000
\end{bmatrix}
\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} 0.578369 \\ 0.000000 \\ 0.000000 \\ 0.000000 \\ 0.000000 \\ 0.000000 \\ 0.421631 \end{bmatrix}.
\end{aligned}
$$

Thus, with probability 57.8% our gambler is broke and with probability 42.2% our gambler has won and now has \$60.00.

Problem 3.2.11 *Suppose the gambler starts with \$20.00. What is the probability that after many spins of the wheel the gambler has won and what is the probability that after many spins of the wheel the gambler has lost?*

Problem 3.2.12 *Suppose the gambler starts with \$40.00. What is the probability that after many spins of the wheel the gambler has won and what is the probability that after many spins of the wheel the gambler has lost?*

Problem 3.2.13 *A chemical is injected into Cell 1 of a rectangular 4-celled organism (below). The molecules of the chemical move randomly between cells that share a side. Each second, the probability that a molecule moves from one cell to an adjacent cell is* $1/5$. *Determine the distribution of the chemical in this organism after 10, 50 and 90 seconds.*

Cell 4	Cell 3
Cell 1	Cell 2